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ON WELL-BEHAVED C*-ALGEBRAS RELATED TO ORDERS

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One of the outstanding problems in the theory of AW*-algebras is the monotone completeness of any AW*-algebras. For AW*-algebras of Type I, the answer is known to be yes (see Kaplansky [3]) but, for general AW*-algebras, this question is still open, although an impressive attack on this problem was made by Christensen and Pedersen [1].

In this note, we should like to make a survey of the development of the problem of monotonicity of AW*-algebras, with an outline of their proofs. This is a joint work with John D.M. Wright [6].

Let us recall that a C*-algebra A is an AW*-algebra if (1) each maximal abelian *-subalgebra of A is generated by its projections and (2) each orthogonal family of projections $\{ e_\alpha \}$ in A has a supremum $\Sigma_A e_\alpha$ in $\mathrm{Proj}(A)$ (the complete lattice of all projections in A).

A natural line of attack on this problem would be to use the second dual A'' or the weak closures of representatives of A on some Hilbert spaces.

Unfortunately, this is too naive. In general, the structure of the complete lattice $\mathrm{Proj}(A)$ is not consistent with that of $\mathrm{Proj}(A'')$ or that of the weak closures, because of the lack of the weak or strong topologies. In fact, if so, A would be a von Neumann algebra.

Let B be an AW*-algebra and let C be a unital C*-subalgebra of B. We say that C is _normal in_ B if for every _orthogonal_ family $\{ e_\alpha \}$ in Proj(C) with the supremum $\Sigma_C e_\alpha$ in Proj(C), $\Sigma_C e_\alpha = \Sigma_B e_\alpha$.

Let A be an AW*-algebra. Then A sits inside its regular completion $\hat{A}$ [2]. $\hat{A}$ is a monotone complete C*-algebra (and so an AW*-algebra) which is, in general, not a von Neumann algebra. We say that A is _normal_ if A is normal in $\hat{A}$. So our first question is this:

Are all AW*-algebras normal ?

It has been known for ten years that finite AW*-algebras are normal [7], [2] and [4]. So, when establishing normality, we may confine our attention to properly infinite AW*-algebras.

Quite recently, we showed that mild restrictions on the centre of an AW*-algebra are sufficint to force it to be normal. In particular, all AW*-factors are normal.

Let A be an AW*-algebra whose centre is locally countably decomposable. Then A must be normal.

Detailed proof will appear in the Journal of the London Mathematical Society (see [6]).

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Let B be a C*-algebra. A net of increasing projections in Proj(B), $\{ e_j \}_{j \varepsilon J}$ with the supremum $LUB_{Proj(B)} e_j$ in Proj(B) is said to be <u>well-behaved</u> if $LUB_{Proj(B)} e_j$ <u>is the supremum of</u> $\{ e_j \}_{j \varepsilon J}$ <u>in the partially ordered set</u> B_h. B is said to be <u>well-behaved</u> if, every such net $\{ e_j \}_{j \varepsilon J}$ is well-behaved.

Let us begin with the following lemma which plays an important role in proving the theorem.

Lemma 1. ([4]) <u>Let</u> A <u>be an</u> AW*<u>-algebra</u>. <u>Then the following three conditions are equivalent</u>.

(1) A <u>is normal</u>;

(2) A <u>is well-behaved</u>;

(3) <u>for every increasing net</u> $\{ e_j \}_{j \varepsilon J}$ <u>in</u> Proj(A) <u>with the supremum</u> e <u>in</u> Proj(A), <u>whenever</u> x, <u>in</u> A_h, <u>satisfies</u> $e_j x e_j \geqq 0$ <u>for all</u> j, <u>then</u> $exe \geqq 0$.

Remark. (2) and (3) are equivalent even when A is a general unital C*-algebra.

<u>Outline of the proof</u> (see [4]).

(1) $\rightleftarrows$ (2). Suppose that A is normal, then, by a result of Pedersen and Saitô, for every increasing net $\{ e_j \}_{j\varepsilon J}$ in $\mathrm{Proj}(A)$,

$$\mathrm{LUB}_{\mathrm{Proj}(A)} e_j = \mathrm{LUB}_{\mathrm{Proj}(\hat{A})} e_j .$$

Since $\hat{A}$ is monotone complete and $\{ e_j \}_{j\varepsilon J}$ is well-behaved in $\hat{A}$, and so it is well-behaved in A as well.

Conversely, suppose that A is well-behaved. Then, for every <u>orthogonal</u> family $\{ e_j \}_{j\varepsilon J}$ in $\mathrm{Proj}(A)$, the net $\{ \Sigma_{j\varepsilon F} e_j \mid F$ a non-empty finite subset of $J \}$ is well-behaved. So it follows that

$$\begin{aligned} \Sigma_A e_j &= \mathrm{LUB}_{\mathrm{Proj}(A)} \{ \Sigma_{j\varepsilon F} e_j \mid F \} \\ &= \mathrm{LUB}_{\mathrm{Proj}(\hat{A})} \{ \Sigma_{j\varepsilon F} e_j \mid F \} \\ &= \Sigma_{\hat{A}} e_j . \end{aligned}$$

(2) $\rightleftarrows$ (3). It is given that $\{ e_j \}_{j\varepsilon J}$ is an increasing net in $\mathrm{Proj}(A)$ with the supremum $\mathrm{LUB}_{\mathrm{Proj}(A)} e_j$ (= e say). Suppose that $\{ e_j \}_{j\varepsilon J}$ satisfies (3).

The claim is that $\mathrm{LUB}_{\mathrm{Proj}(A)} e_j = \mathrm{LUB}_{A_h} e_j$. We have only to check that $e_j \leqq a$ for all j for some a in A_h implies

$e \leqq a$. Suppose that such an a is given as above, then, because $a \geqq 0$, it follows that

$$(a + 1/n)^{-1/2} e_j (a + 1/n)^{-1/2} \leqq 1$$

for each j and n. Thus we get that

$$\|(a + 1/n)^{-1/2} e_j)\| \leqq 1$$

for each j and n. This implies that

$$e_j (e - e(a + 1/n)^{-1} e) e_j \geqq 0$$

for all j and n. Since $\{ e_j \}_{j \varepsilon J}$ satisfies (3), it follows that

$$e(e - e(a + 1/n)^{-1} e) e \geqq 0$$

and $\|e(a + 1/n)^{-1} e\| \leqq 1$ for all n. Thus we conclude that

$$(a + 1/n)^{-1/2} e (a + 1/n)^{-1/2} \leqq 1$$

for all n. This implies that $e \leqq a + 1/n$ for all n and so $e \leqq a$ follows.

Conversely suppose that $\{ e_j \}_{j \varepsilon J}$ is well-behaved. It is given $x \varepsilon A_h$ such that $e_j x e_j \geqq 0$ for all j. To prove the claim, we may assume that $e = 1$(consider it in eAe) and $\|x\| \leqq 1$. Since

$$(1 + x)(1 - e_j)(1 + x) - (1 - x)(1 - e_j)(1 - x)$$
$$= 2x(1 - e_j) + 2(1 - e_j)x,$$

we see that

$$e_jxe_j - x = (1 - e_j)x(1 - e_j) - (1 - e_j)x - x(1 - e_j)$$

$$= (1/2)((1 - x)(1 - e_j)(1 - x)$$

$$- (1 + x)(1 - e_j)(1 + x)) + (1 - e_j)x(1 - e_j)$$

$$\leqq (1/2)(1 - x)(1 - e_j)(1 - x) + 1 - e_j$$

because $\|x\| \leqq 1$ and $x = x^*$. Take $y = (1/2)(|x| + x)$ and $z = (1/2)(|x| - x)$. We see that $x = y-z$, y,z in A_h, $yz = 0$ and z and y are non-negative. Moreover, y and z commute with x. Hence, it follows that

$$ze_jxe_jz - zxz \leqq (1/2)z(1 - x)(1 - e_j)(1 - x)z + z(1 - e_j)z.$$

Since ze_jxe_jz is non-negative for all j and $zxz = - z^3$, we see that

$$z^3 \leqq (1/2)z(1 - x)(1 - e_j)(1 - x)z + z(1 - e_j)z$$

for all j. Since $\{ e_j \}_{j\varepsilon J}$ is well-behaved, this implies that

$$(1/2)z(1 - x)(1 - e_j)(1 - x)z + z(1 - e_j)z \downarrow 0 \text{ in } A_h,$$

and so $z^3 = 0$, that is, $z = 0$. This completes the proof.

Theorem 1. Finite AW*-algebras are normal.

Let $\{ e_j \}_{j\varepsilon J}$ be any increasing net in Proj(A) with the supremum e in Proj(A). We shall show that $\{ e_j \}_{j\varepsilon J}$ satisfies (3). To do this, we may assume that $e = 1$. Suppose that x in A_h satisfies that $e_j x e_j \geqq 0$ for all j. If $x = x^+ - x^-$ ($x^+x^- = 0$, $x^+ \geqq 0$ and $x^- \geqq 0$) and $x^- \neq 0$, then there is a non-zero projection q in A and a positive number ε such that $x^- \geqq \varepsilon q$ and $(1 - q)x^+ = x^+$. Set $f_j = e_j \wedge q$ and we have

$$0 \leqq f_j e_j x e_j f_j = f_j x f_j = f_j q x q f_j = - f_j x^- q f_j$$

$$\leqq -\varepsilon f_j q f_j \leqq -\varepsilon f_j$$

for all j and so $f_j = 0$ for all j, that is, $e_j \wedge q = 0$ for all j. Note that

$$q = q - e_j \wedge q \sim e_j \vee q - e_j \leqq 1 - e_j$$

for all j and A is finite, this implies that $q = 0$, because $1 - e_j \downarrow 0$ in Proj(A). This is a contradiction. Thus $x^- = 0$, that is $x \geqq 0$. This completes the proof.

Now we are in the position to discuss about the properly infinite case. Since, as you see, above proof depends on the finiteness assumption on A, we need to seek another way to establish the normality for properly infinite case.

Before going into the discussions, we need some definitions.

For a given index set β, a family $\{ x_j \varepsilon A_h \mid j \varepsilon \beta \}$ is said to be <u>order summable</u> if

$$\{ \Sigma_{j \varepsilon F} x_j \mid F \text{ a non-empty finite subset of } \beta \}$$

is bounded above in A_h. When such a family is order summable its <u>order sum</u> is defined to be the supremum of the set

$$\{ \| \Sigma_{j \varepsilon F} x_j \| \mid F \text{ a non-empty finite subset of } \beta \}.$$

For a given indexed set β, a family $\{ x_j \varepsilon A_h \mid j \varepsilon \beta \}$ is said to be <u>well-behaved</u> if the set

$$\{ \Sigma_{j \varepsilon F} x_j \mid F \text{ a non-empty finite subset of } \beta \}$$

has a supremum in A_h. When $\{ x_j \mid j \varepsilon \beta \}$ is a family of orthogonal projections, this definition of well-behaved is consistent with our earlier one.

Let β be a given index set. The algebra A is said to be β-<u>complete</u> if each order summable, β-indexed family of positive elements in A is well-behaved.

It is clear that if A is β-complete and if γ is a set where $\# \gamma \leqq \# \beta$, then A is γ-complete. We note further that A is α-complete for a sufficiently large ordinal α, then A is monotone complete, but we omit the details.

The rest of the discussions, we shall suppose that A is a <u>properly infinite</u> AW*-<u>algebra</u>.

We suppose for the moment that, for <u>some infinite ordinal</u> Ω, A has a system of matrix units $\{ e_{ij} \}_{0 \leqq i,j < \Omega}$ where

$e_{00} \sim 1$ in A.

A transfinite sequence $\{ a_j \}_{j<\alpha}$ in $e_{00}Ae_{00}$ is said to be <u>dilatable</u> in A if there exists an orthogonal family of projections $\{ p_j \}_{j<\alpha}$ in $(\Sigma_{j<\alpha}e_{jj})A(\Sigma_{j<\alpha}e_{jj})$ such that $e_{00}p_je_{00} = x_j$ for each $j < \alpha$. Let α be an ordinal number. We call $e_{00}Ae_{00}$ <u>α-dilatable</u> if, whenever $\{ x_j \}_{j<\alpha}$ is an order summable transfinite sequence of positive elements of $e_{00}Ae_{00}$, with order sum <u>less than</u> 1, then the transfinite sequence is dilatable in A.

The following lemma is a modification of an ingenious argument by Christensen and Pedersen [1].

Lemma 3. <u>Let α be an ordinal with $\alpha \leqq \Omega$. Let $e_{00}Ae_{00}$ be α-dilatable in A. Then A is α-complete.</u>

Since $e_{00}Ae_{00}$ is *-isomorphic to A, it suffices to show that $e_{00}Ae_{00}$ is α-complete. The proof is rather long. We shall omit the details. See [6].

We shall need the following lemma which is proved in [1, Lemma 3].

Lemma 4. <u>Let e and p be projections in a unital C*-algebra B such that $\|epe\| < 1$ and let x be a positive element of B such that $x + epe \leqq e$. Let $\{ f_{ij} \}_{1\leqq i,j\leqq 2}$ be matrix units</u>

for $M_2(C)$. *Then there exists a projection* q *in* $B \otimes M_2(C)$ *such that* q *is orthogonal to* $p \otimes f_{11}$ *and*

$$(e \otimes f_{11}) q (e \otimes f_{11}) = x \otimes f_{11}.$$

Let $z = (1 - p)(1 - epe)^{-1} x (1 - epe)^{-1}(1 - p)$. Then $z \in A_h$ such that $(1 - p)z(1 - p) = z$, $eze = x$ and $0 \leqq z \leqq 1$. Let

$$q = \begin{pmatrix} z & (z-z^2)^{1/2} \\ (z-z^2)^{1/2} & 1 - z \end{pmatrix}$$

via $\{ f_{ij} \}_{1 \leqq i,j \leqq 2}$. Then q satisfies all the requirements.

Lemma 5. *Let* $\alpha < \Omega$ *and let* $\alpha + 1$ *be the successor ordinal of* α. *Let* $e_{00}Ae_{00}$ *be* α*-dilatable in* A. *Then* $e_{00}Ae_{00}$ *is also* $(\alpha + 1)$*-dilatable*.

In fact, let $\{ x_\xi \}_{\xi<\alpha+1}$ be an order summable transfinite sequence of positive elements of $e_{00}Ae_{00}$. Let its order sum be c, where $c < 1$. By hypothesis, there exists a family of orthogonal projections $\{ p_\xi \}_{\xi<\alpha}$ in $(\Sigma_{i<\alpha} e_{ii}) A (\Sigma_{i<\alpha} e_{ii})$ such that $e_{00} p_\xi e_{00} = x_\xi$ for each $\xi < \alpha$. By Lemma 3, A is α-complete and so

$$\Sigma_{i<\alpha} p_i = \mathrm{LUB}_{A_h} \{ \Sigma_{i \in F} p_i \mid F \text{ a non-empty finite subset of } \alpha \}$$

Thus

$$e_{00}(\Sigma_{i<\alpha}p_i)e_{00} = \mathrm{LUB}_{A_h}\{ \Sigma_{i\varepsilon F}x_i \mid F \text{ a finite subset of } \alpha\}$$

and so $\| e_{00}(\Sigma_{i<\alpha}p_i)e_{00}\| \leqq c < 1$. Also

$$e_{00}(\Sigma_{i<\alpha}p_i)e_{00} + x_\alpha \leqq ce_{00} < e_{00}.$$

We observe that $\Sigma_{i<\alpha}e_{ii} \sim e_{\alpha\alpha} \sim \Sigma_{i\leqq\alpha}e_{ii}$. Let $f_{11} = \Sigma_{i<\alpha}e_{ii}$, let $f_{22} = e_{\alpha\alpha}$ and let f_{12} be any partial isometry in $(\Sigma_{i\leqq\alpha}e_{ii})A(\Sigma_{i\leqq\alpha}e_{ii})$ such that $f_{12}f_{12}{}^* = f_{11}$ and $f_{12}{}^*f_{12} = f_{22}$. Let $f_{21} = f_{12}{}^*$. Then, by the above lemma, there is a projecton p_α in $(\Sigma_{i\leqq\alpha}e_{ii})A(\Sigma_{i\leqq\alpha}e_{ii})$, such that p_α is orthogonal to $\Sigma_{i<\alpha}p_i$ and $x_\alpha = e_{00}p_\alpha e_{00}$. Hence $\{ x_\xi \mid \xi \leqq \alpha \} = \{ x_\xi \mid \xi < \alpha + 1 \}$ is dilatable.

Lemma 6. <u>Let α be an infinite ordinal such that $\alpha \leqq \Omega$. Let A be ξ-complete for each $\xi < \alpha$. Then $e_{00}Ae_{00}$ is α-dilatable in A.</u>

In fact, let $\{ x_i \}_{i<\alpha}$ be an order summable transfinite sequence of positive elements of $e_{00}Ae_{00}$ with order sum c, where $c < 1$. To obtain a contradiction, let us assume that this transfinite sequence is not dilatable. Then, there exists a smallest ordinal β, such that $\{ x_i \}_{i<\beta}$ is not dilatable, and $\beta \leqq \alpha$. (Note that $\omega < \beta$ by the results of Christensen and Pedersen [1].)

Let γ be any non-zero ordinal strictly less than β. From the definition of β, $\{ x_i \}_{i<\gamma}$ is dilatable. So, there is a family $\{ p_i \}_{i<\gamma}$ of orthogonal projections in $(\Sigma_{i<\gamma}e_{ii})A(\Sigma_{i<\gamma}e_{ii})$ such that $x_i = e_{00}p_ie_{00}$ for each $i < \gamma$. Let M_γ be the set of all such families of orthogonal projections and let $M = \cup\{ M_\gamma \mid 0 < \gamma < \beta \}$. Clearly $M \neq \emptyset$. For $\Gamma = \{ p_i \}_{i<\gamma_1} \varepsilon M$ and $\Xi = \{ q_j \}_{j<\gamma_2} \varepsilon M$, we define $\Gamma \leqq \Xi$ to mean that $\gamma_1 \leqq \gamma_2$ and, for all $i < \gamma_1$, $p_i = q_i$. This partially orders M <u>inductively</u>. So, by Zorn's lemma, M has a maximal element $\{ p_i \}_{i<\zeta}$. Then, by applying the argument of Lemma 5 to $\{ x_\xi \}_{\xi<\zeta+1}$, we find a projection p_ζ in $(\Sigma_{i\leqq\zeta}e_{ii})A(\Sigma_{i\leqq\zeta}e_{ii})$ such that p_ζ is orthogonal to $\Sigma_{i<\zeta}p_i$ and $e_{00}p_\zeta e_{00} = x_\zeta$. Thus $\{ p_i \}_{i<\zeta+1}$ is in M. This contradicts maximality. Hence the assumption that $\{ x_i \}_{i<\alpha}$ was not dilatable must be false. Hence $e_{00}Ae_{00}$ is α-dilatable in A.

By using these lemmas, we have the following:

Theorem 2. <u>Let</u> A <u>be a properly infinite</u> AW*-<u>algebra</u>. <u>Let</u> Ω <u>be an infinite ordinal such that there exists an</u> Ω-<u>indexed system of matrix units in</u> A, $\{ e_{ij} \}_{0\leqq i,j<\Omega}$. <u>Then</u> A <u>is</u> Ω-<u>complete</u>.

In fact, assume that A is not Ω-complete. Then there is a first ordinal β, $\beta \leqq \Omega$, such that A is not β-complete. So, for $\alpha < \beta$, A is α-complete. So, by the above lemma,

$e_{00}Ae_{00}$ is β-dilatable in A. Then, by Lemma 3, A is β-complete. This is a contradiction. So A must be Ω-complete.

Corollary 1([1]). *Let* A *be a properly infinite* AW*-*algebra*. *Then* A *is monotone* σ-*complete*.

Since A has a countable system of matrix units, A is ω-complete. So, A is monotone σ-complete.

Now we are in the position to discuss about normality in properly infinite AW*-algebras.

Theorem 3. *Let* A *be a properly infinite* AW**-algebra whose centre*, Z, *is locally countably decomposable. Then* A *is normal*.

Outline of the proof. (See [6].)

Let A be an infinite AW*-*factor*. Let Π be an infinite set of orthogonal projections in A. Let $\aleph$ be the cardinalit; of Π. Then A is $\aleph$-complete. Since A is monotone σ-complete, there is nothing further to prove if Π is countable. So let us suppose Π to be uncountably infinite.

We may decompose Π into a family of disjoint set $\{ \Pi_\lambda \mid \lambda \in \Lambda \}$, where each Π_λ is of the same cardinality as Π, and where $\#\Pi = \#\Lambda$. Let $p_\lambda = \Sigma\Pi_\lambda$. Then $\{ p_\lambda \mid \lambda \in \Lambda \}$ is an orthogonal family of non-zero projections. We shall show that each p_i is infinite. Suppose that p_i is finite for some i. Then $p_i A p_i$ is a finite AW*-factor and so it is σ-finite. Since Π_i is uncountable. This is a contradiction. So all p_i are infinite. Since A is infinite, there exists a minimal infinite projection e_0 in A such that $e_0 \precsim p_i$ for all $i \in \Lambda$. So there is a set Π' of mutually orthogonal family of projections $\{ e_\lambda \mid \lambda \in \Lambda \}$ such that $e_\lambda \sim e_0$ for all $\lambda \in \Lambda$. By Zorn's lemma, Π' can be extended to a maximal collection Γ of mutually orthogonal infinite projections, each of which is equivalent to e_0. Clearly $\#\Gamma \geqq \aleph$. By a general theory of AW*-algebras, one can find a $\#\Gamma$-homogeneous partition of 1 in A. Hence we can construct $\#\Gamma$-system of matrix units $\{ e_{ij} \mid ij \in \Gamma \}$ in A such that $e_{ii} \sim 1$ for all i. Since $\#\Lambda \leqq \#\Gamma$, A must be $\#\Lambda$-complete, and so Π is well-behaved. For the general case, see [6].

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